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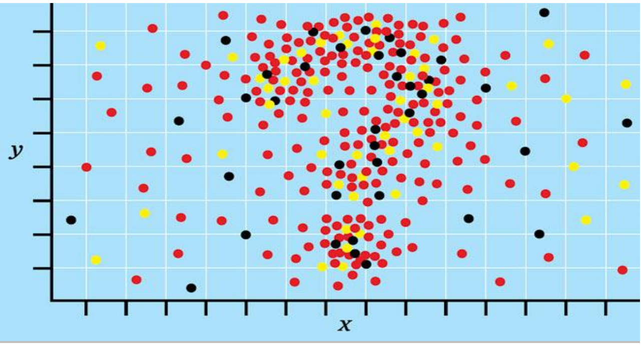

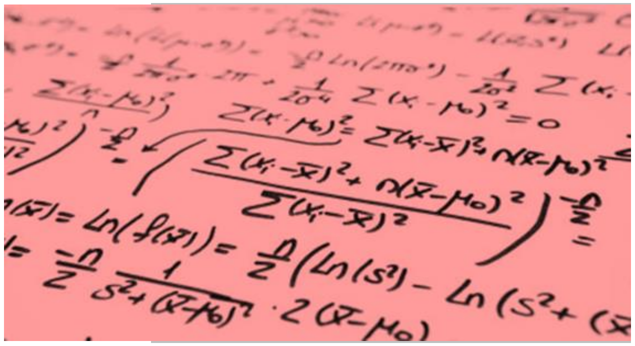

Comparison of ALT Plans Based on Exact Small-Sample Methodology and Asymptotic Large-Sample Methodology

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Outline

- Motivation
- Models, Notation, and Optimization Criteria
- Small-Sample and Large-Sample Methodology
- Test Plan Comparisons
- Sample Size Selection
- Summary and Conclusions

Motivation

- Much of the statistical literature on optimal accelerated test planning uses methodology based on large-sample methods.
  - Assumes sufficient sample size or lifetime distribution.
  - Results in an approximation of the true variance.
- These methods may not yield the desired precision under small-sample settings.
  - May underestimate the true precision, leading to a false sense of confidence.
- Determining the exact variance is difficult due to the often intractable mathematics involved.

Purpose is to create methodology for developing optimal test plans that are a function of the sample size.

Models and Notation – Log-Location Scale Family

- Let t_{ij} be the observed time to failure of unit $i = 1,2, \dots, n_j$ tested at stress level $j = 1,2, \dots, J$.
- A distribution is a member of the *log-location scale* family if it's density function is of the form

$$f(t_{ij}) = \frac{1}{\sigma t_{ij}} \phi\left(\frac{\ln(t_{ij}) - \mu_j}{\sigma}\right)$$

  - The location parameter μ_j will vary across stress levels while the scale parameter σ will be assumed constant across all stress levels.
- An alternative form for these distributions is

$$g(t_{ij}) = \frac{k}{\lambda_j}\left(\frac{t_{ij}}{\lambda_j}\right)^{k-1} \psi\left[\left(\frac{t_{ij}}{\lambda_j}\right)^{k}\right]$$

  - The transformation between the two forms is $\lambda_j = e^{\mu_j}$ and $k = {}^{1}/_{\sigma}$.

Models and Notation – Log-Location Scale Family

- The most popular members of this family are the lognormal and Weibull distributions.

- Let $z = \frac{\ln(t_{ij})-\mu_j}{\sigma}$ and $\omega = \left(\frac{t_{ij}}{\lambda_j}\right)^k$

Distribution	$\phi(z)$	$\psi(\omega)$
Lognormal	$\frac{1}{\sqrt{2\pi}} e^{\left(-\frac{z^2}{2}\right)}$	$\frac{1}{\sqrt{2\pi}} \omega^{-(1+0.5 \ln \omega)}$
Weibull	$e^{(z-e^z)}$	$e^{-\omega}$

Models and Notation – Log-Location Scale Family

- The quantile function for a log-location scale family member is given as
  - $t_{pj} = e^{\mu_j + \sigma z_p}$ for the form $f(t_{ij})$, where z_p is the pth quantile of the standard form
  - $t_{pj} = \lambda_j \omega_p^{1/k}$ for the form $g(t_{ij})$, where ω_p is the pth quantile of the standard form
    - This is the quantile function we will focus on.

Distribution	z_p	ω_p
Lognormal	$\Phi^{-1}(p)$	$e^{\Phi^{-1}(p)}$
Weibull	$\ln(-\ln(1-p))$	$-\ln(1-p)$

Models and Notation – Acceleration Models

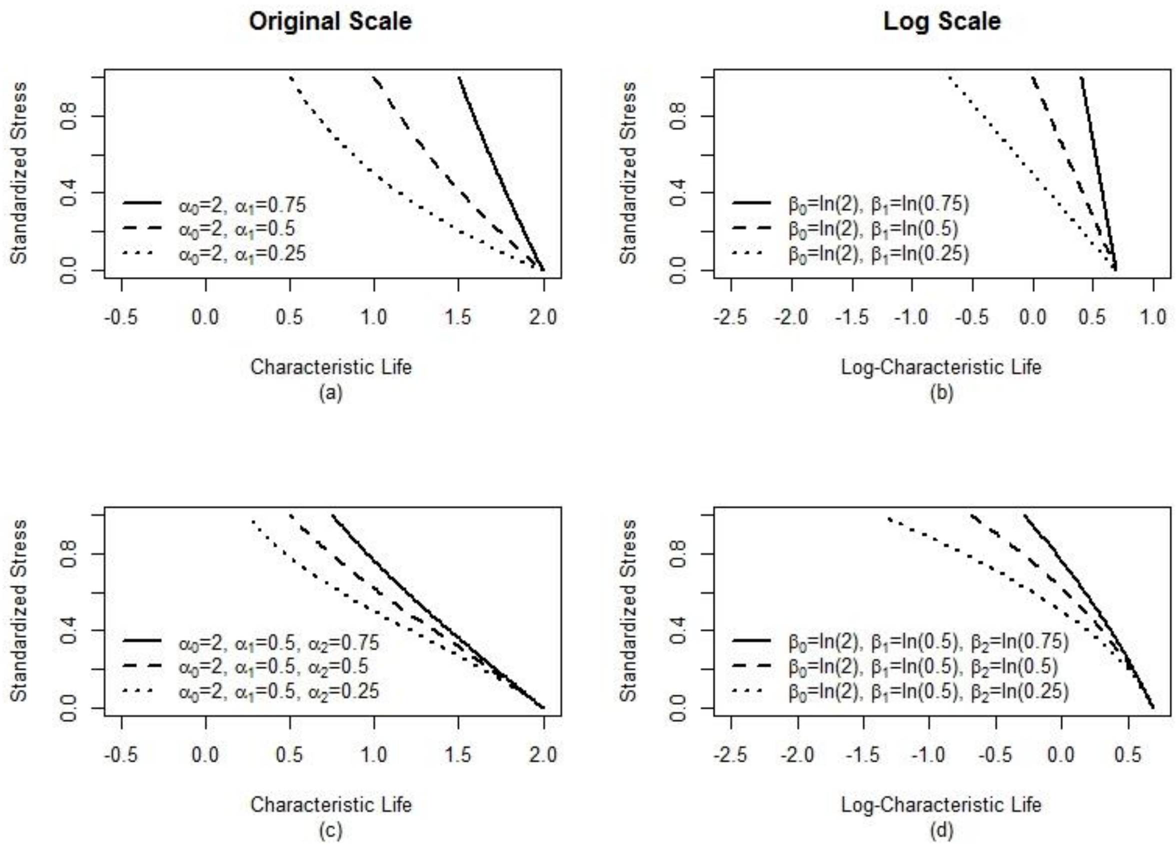
- Let $x_j, j = 0,1,2, \ldots, J$ represent the value of stress level j.
  - $j = 0$ represents the use condition
  - $x_0 < x_1 < x_2 < \cdots < x_J$
  - The values are transformed such that $x_0 = 0$ and $x_J = 1$.
- Let $n_j, j = 1,2,..,J$ represent the number of samples allocated to stress level j.
  - The total sample size $N = \sum_{j=1}^{J} n_j$
- Two acceleration models are considered:

Model	λ_j	μ_j
Exponential Model (EM)	$\lambda_j = \alpha_0 \alpha_1^{x_j}$	$\mu_j = \beta_0 + \beta_1 x_j$
Quadratic Exponential Model (QEM)	$\lambda_j = \alpha_0 \alpha_1^{x_j} \alpha_2^{x_j^2}$	$\mu_j = \beta_0 + \beta_1 x_j + \beta_2 x_j^2$

Models and Notation – Acceleration Models

EM

QEM



Optimization Criteria

Given a range and number of design points ($\boldsymbol{x}$), optimize the location of the design points and sample allocation ($\boldsymbol{n}$) to each point to yield minimum variance of the *p*th quantile estimator $\hat{t}_p$ at the use condition $x_0 = 0$.

- Use *p*=0.9 and $x_1 = 0.1$ for illustration.

- Asymptotic Approach
  - $\min\limits_{\boldsymbol{x},\boldsymbol{n}} AVar\left(\log \hat{t}_p\right) = AVar\left(\hat{\beta}_0 + \tilde{\sigma} z_p\right)$
    - $\tilde{\sigma}$ either known or estimated; z_p is the *p*-quantile of the standard form
- Small-Sample Approach
  - $\min\limits_{\boldsymbol{x},\boldsymbol{n}} Var\left(\hat{t}_p\right) = Var\left(\hat{\alpha}_0 \omega_p^{1/\tilde{k}}\right)$
    - $\tilde{k}$ either known or estimated; ω_p is the *p*-quantile of the standard form
- Where possible, the maximum likelihood (ML) estimator is used.
- The number of design points is at least the number of unknown model parameters.

Small-Sample Method – Estimator Derivation

1. Start with $\lambda_{M,j} = \alpha_0 f(x_j; \boldsymbol{\alpha})$ for specified model *M*
2. Set up a system of *d* equations in *d* unknown parameters and solve for α_0 as a function of $\lambda_{M,j}$.
   - Other parameters may be irreducibly involved as well.
3. Determine a suitable estimator for $\lambda_{M,j}$ and any other parameters.
   - $\hat{\lambda}_{M,j} = \prod_{i=1}^{n_j} t_{ij}^{1/n_j}$ for lognormal; $\hat{\lambda}_{M,j} = \left(\frac{1}{n_j}\sum_{i=1}^{n_j} t_{ij}^k\right)^{1/k}$ for Weibull
4. Substitute in the estimators to yield the estimator for α_0.

- The estimation procedure yields the ML estimator under certain conditions.
  - Otherwise, referred to as Near-Exact (NE) estimator as it yields near-exact variance of ML estimator in specific settings.

Small-Sample Method – Estimator Derivation Example

- For $\lambda_j = \alpha_0 \alpha_1^{x_j}$,
  - $\alpha_1 = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{x_2 - x_1}}$; $\alpha_0 = \lambda_1^{\varepsilon_{21}} \lambda_2^{\varepsilon_{12}}$
  - $\varepsilon_{ij} = \frac{x_i}{x_i - x_j}$, throughout

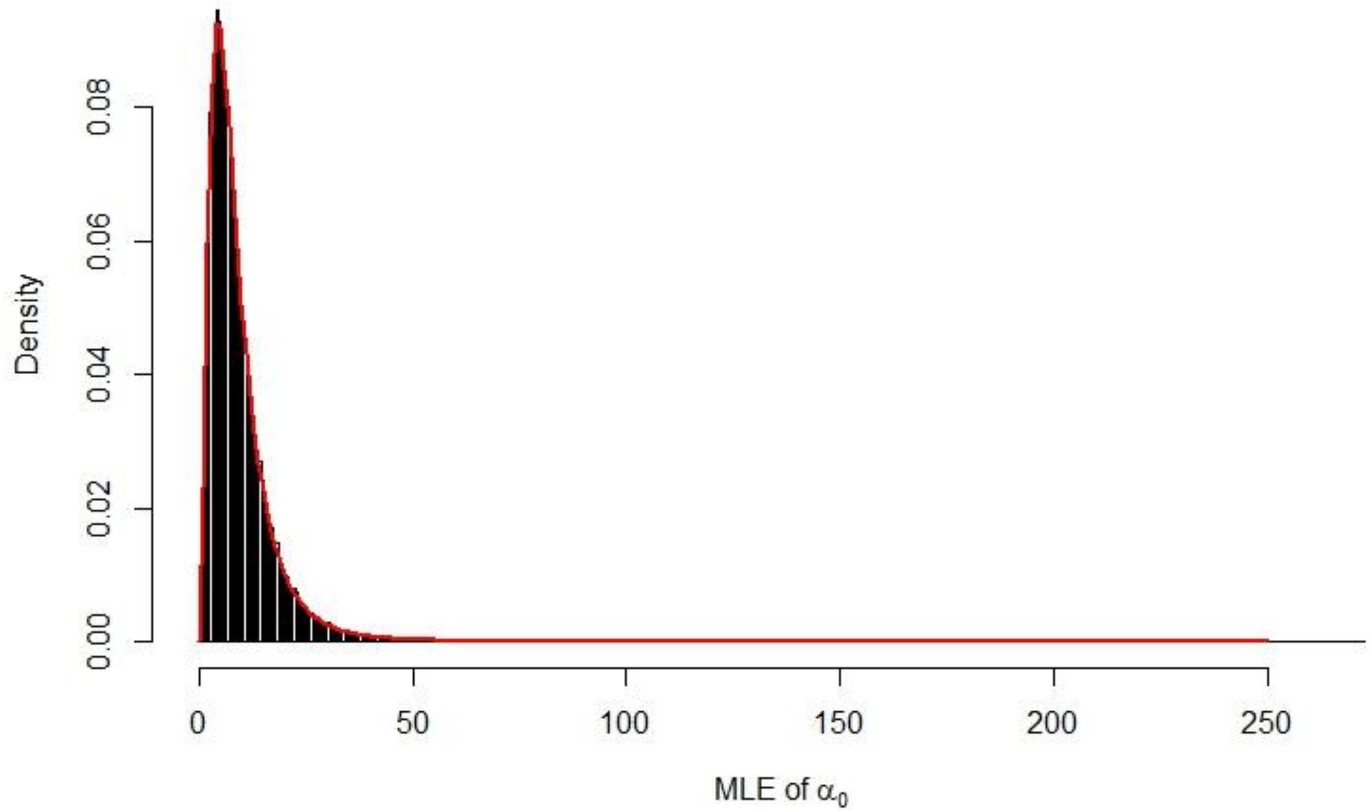
Distribution	$\widehat{\boldsymbol{\alpha}}_{\mathbf{0}}$
Lognormal	$\widehat{\boldsymbol{\alpha}}_{\mathbf{0}} = \left(\prod_{i=1}^{n_1} t_{i1}^{1/n_1}\right)^{\varepsilon_{21}} \left(\prod_{i=1}^{n_2} t_{i2}^{1/n_2}\right)^{\varepsilon_{12}}$
Weibull	$\widehat{\boldsymbol{\alpha}}_{\mathbf{0}} = \left(\frac{1}{n_1}\sum_{i=1}^{n_1} t_{i1}^{k}\right)^{\varepsilon_{21}/k} \left(\frac{1}{n_2}\sum_{i=1}^{n_2} t_{i2}^{k}\right)^{\varepsilon_{12}/k}$

Small-Sample Method – NEE Variance

Simulations Based on QEM; Weibull Lifetime Distribution, Shape Parameter Known

Model Parameters			Design Points			Sample Allocation			Shape Parameter	MLE Simulated Variance	NEE Exact Variance
β_0	β_1	β_2	x_1	x_2	x_3	n_1	n_2	n_3			
2	-0.5	0.75	0.1	0.55	1	5	2	3	2	8.9504	8.9835
2	-0.5	0.75	0.1	0.55	1	4	3	3	2	8.4315	8.4249
2	-0.5	0.75	0.1	0.55	1	6	2	2	2	8.0119	8.0210
2	-0.5	0.75	0.1	0.65	1	5	2	3	2	9.1159	9.1459
2	-0.5	0.75	0.1	0.45	1	5	2	3	2	10.1036	10.1388
2	-0.5	0.75	0.1	0.55	1	5	2	3	1	66.8808	66.9007
2	-0.5	0.75	0.1	0.55	1	5	2	3	3	3.6038	3.6196
1	-0.5	0.75	0.1	0.55	1	5	2	3	2	0.4877	0.4898
3	-0.5	0.75	0.1	0.55	1	5	2	3	2	26.6288	26.7457

Small-Sample Method – QEM



Small-Sample Method – Variance Derivation

- When the shape parameter is known, the variance of the quantile estimator is driven solely by the variance of the estimator $\widehat{\boldsymbol{\alpha}}_{\mathbf{0}}$.
- When the shape parameter is unknown, the variance of the quantile estimator can be assessed using Monte Carlo integration.
  - Necessary for the Weibull distribution; optional for the lognormal, due to a closed yet complex form available.

Small-Sample Method – Variance Derivation

Shape Parameter Known

- Exponential Model ($\hat{\alpha}_0 = \hat{\lambda}_1^{\varepsilon_{21}} \hat{\lambda}_2^{\varepsilon_{12}}$)

Distribution	Var($\hat{\alpha}_0$)
Lognormal	$\alpha_0^2 \left\{ \exp\left[\frac{1}{k^2} \left(\frac{\varepsilon_{21}^2}{n_1} + \frac{\varepsilon_{12}^2}{n_2} \right) \right] - 1 \right\} \exp\left[\frac{1}{k^2} \left(\frac{\varepsilon_{21}^2}{n_1} + \frac{\varepsilon_{12}^2}{n_2} \right) \right]$
Weibull	$\alpha_0^2 \left(\frac{1}{n_1^{\varepsilon_{21}} n_2^{\varepsilon_{12}}} \right)^{2/k} \left\{ \frac{\Gamma\left(n_1 + \frac{2\varepsilon_{21}}{k}\right) \Gamma\left(n_2 + \frac{2\varepsilon_{12}}{k}\right)}{\Gamma(n_1)\Gamma(n_2)} - \left[\frac{\Gamma\left(n_1 + \frac{\varepsilon_{21}}{k}\right) \Gamma\left(n_2 + \frac{\varepsilon_{12}}{k}\right)}{\Gamma(n_1)\Gamma(n_2)} \right]^2 \right\}$

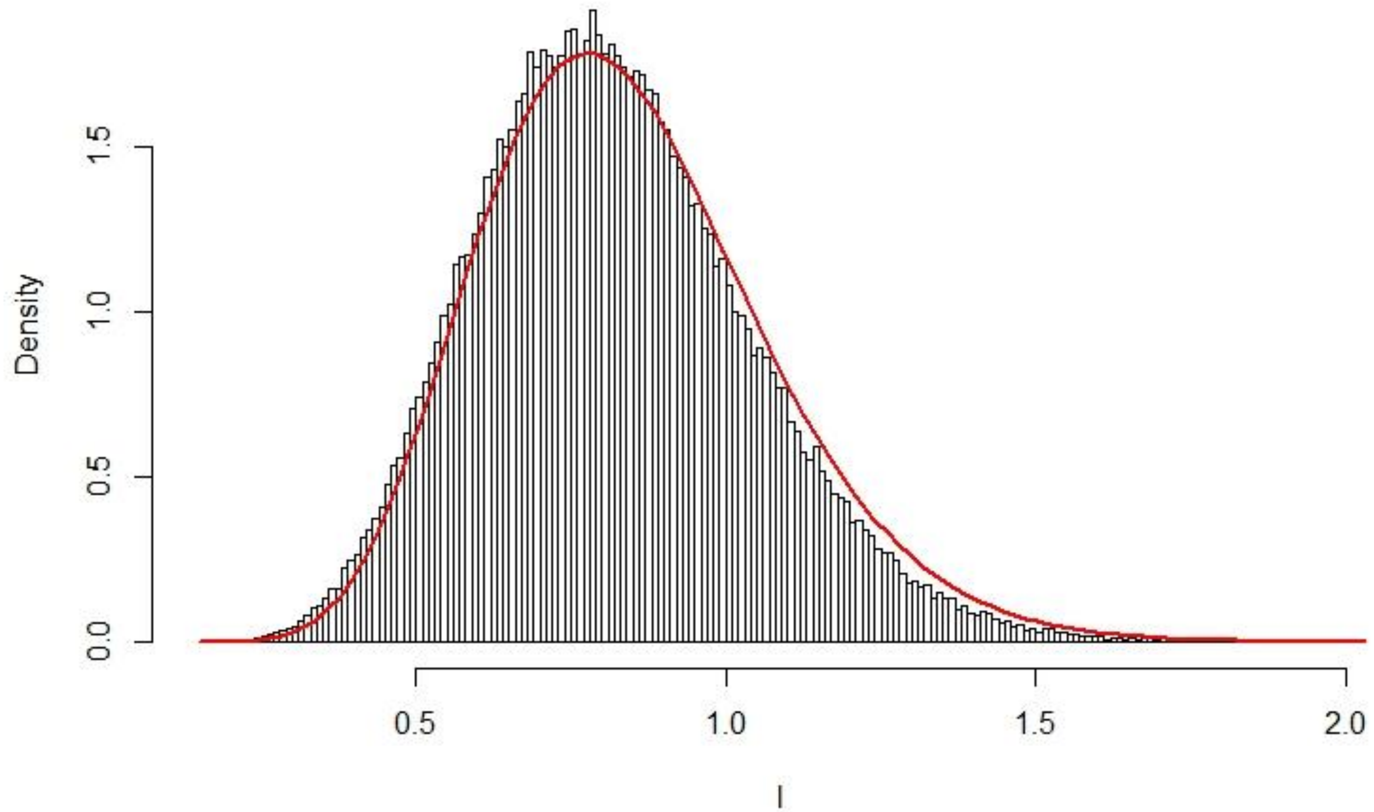
- Quadratic Exponential Model ($\hat{\alpha}_0 = \hat{\lambda}_1^{\varepsilon_{21}\varepsilon_{31}} \hat{\lambda}_2^{\varepsilon_{12}\varepsilon_{32}} \hat{\lambda}_3^{\varepsilon_{13}\varepsilon_{23}}$)

Distribution	Var($\hat{\alpha}_0$)
Lognormal	$\alpha_0^2 \left\{ \exp\left[\frac{1}{k^2} \left(\frac{\varepsilon_{21}^2\varepsilon_{31}^2}{n_1} + \frac{\varepsilon_{12}^2\varepsilon_{32}^2}{n_2} + \frac{\varepsilon_{13}^2\varepsilon_{23}^2}{n_3} \right) \right] - 1 \right\} \exp\left[\frac{1}{k^2} \left(\frac{\varepsilon_{21}^2\varepsilon_{31}^2}{n_1} + \frac{\varepsilon_{12}^2\varepsilon_{32}^2}{n_2} + \frac{\varepsilon_{13}^2\varepsilon_{23}^2}{n_3} \right) \right]$
Weibull	$\alpha_0^2 \left(\frac{1}{n_1^{\varepsilon_{21}} n_2^{\varepsilon_{12}}} \right)^{2/k} \left\{ \frac{\Gamma\left(n_1 + \frac{2\varepsilon_{21}\varepsilon_{31}}{k}\right) \Gamma\left(n_2 + \frac{2\varepsilon_{12}\varepsilon_{32}}{k}\right) \Gamma\left(n_3 + \frac{2\varepsilon_{13}\varepsilon_{23}}{k}\right)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} - \left[\frac{\Gamma\left(n_1 + \frac{\varepsilon_{21}\varepsilon_{31}}{k}\right) \Gamma\left(n_2 + \frac{\varepsilon_{12}\varepsilon_{32}}{k}\right) \Gamma\left(n_3 + \frac{\varepsilon_{13}\varepsilon_{23}}{k}\right)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} \right]^2 \right\}$

Small-Sample Method – Weibull Shape Parameter ML Estimator

- For the Weibull distribution with unknown shape parameter, the shape parameter ML estimator does not have a closed form.
- It is known in the literature that the quantity ${}^{1}/_{l} = {}^{\hat{k}}/_{k}$ has a distribution that does not depend on any unknown quantities.
  - ${}^{k}/_{l}$ is used in place of *k* in the point estimators of $\tilde{\lambda}$
- Simulations show that the quantity $l = {}^{k}/_{\hat{k}}$ is very well approximated by a gamma distribution with shape parameter $\frac{6(N-I)-1}{4}$ and scale parameter $\frac{3(2N-I)}{4}$, where *N* is the total number of samples and *I* is the number of design points.
  - Used in the Monte Carlo integration.

Small-Sample Method – Weibull Shape Parameter ML Estimator



Large-Sample Method – Asymptotic Variance Derivation

1. Start with $\mu_{M,j} = \beta_0 + f(x_j; \boldsymbol{\beta})$ for specified model M
2. Compute the Fisher Information Matrix $\boldsymbol{I}(\beta_0, \boldsymbol{\beta}', \sigma)$.
3. Compute the asymptotic variance as

$$AVar(\hat{t}_p) = (1 \quad \mathbf{0}' \quad z_p)\boldsymbol{I}^{-1}(\beta_0, \boldsymbol{\beta}', \sigma)(1 \quad \mathbf{0}' \quad z_p)'$$

Model	$AVar(\hat{t}_p)$
EM	$\sigma^2 \left\{ \frac{\varepsilon_{21}^2}{n_1} + \frac{\varepsilon_{12}^2}{n_2} + (quadratic\ in\ z_p) \right\}$
QEM	$\sigma^2 \left\{ \frac{\varepsilon_{21}^2 \varepsilon_{31}^2}{n_1} + \frac{\varepsilon_{12}^2 \varepsilon_{32}^2}{n_2} + \frac{\varepsilon_{13}^2 \varepsilon_{23}^2}{n_3} + (quadratic\ in\ z_p) \right\}$

Distribution	Var($\hat{\alpha}_0$)
Lognormal - EM	$\alpha_0^2 \left\{ \exp\left[\frac{1}{k^2} \left(\frac{\varepsilon_{21}^2}{n_1} + \frac{\varepsilon_{12}^2}{n_2} \right) \right] - 1 \right\} \exp\left[\frac{1}{k^2} \left(\frac{\varepsilon_{21}^2}{n_1} + \frac{\varepsilon_{12}^2}{n_2} \right) \right]$
Lognormal - QEM	$\alpha_0^2 \left\{ \exp\left[\frac{1}{k^2} \left(\frac{\varepsilon_{21}^2 \varepsilon_{31}^2}{n_1} + \frac{\varepsilon_{12}^2 \varepsilon_{32}^2}{n_2} + \frac{\varepsilon_{13}^2 \varepsilon_{23}^2}{n_3} \right) \right] - 1 \right\} \exp\left[\frac{1}{k^2} \left(\frac{\varepsilon_{21}^2 \varepsilon_{31}^2}{n_1} + \frac{\varepsilon_{12}^2 \varepsilon_{32}^2}{n_2} + \frac{\varepsilon_{13}^2 \varepsilon_{23}^2}{n_3} \right) \right]$

Large-Sample Method – Asymptotic Variance Derivation

1. Start with $\mu_{M,j} = \beta_0 + f(x_j; \boldsymbol{\beta})$ for specified model M
2. Compute the Fisher Information Matrix $\boldsymbol{I}(\beta_0, \boldsymbol{\beta}', \sigma)$.
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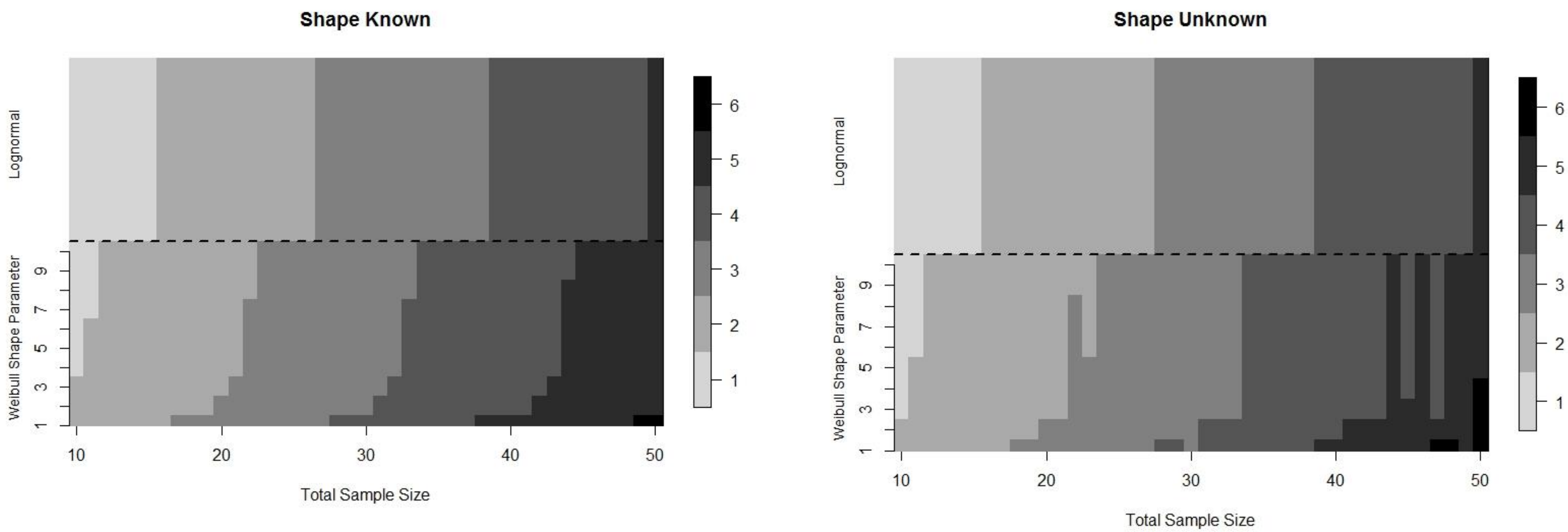
$$AVar(\hat{t}_p) = (1 \quad \mathbf{0}' \quad z_p)\boldsymbol{I}^{-1}(\beta_0, \boldsymbol{\beta}', \sigma)(1 \quad \mathbf{0}' \quad z_p)'$$

Model	$AVar(\hat{t}_p)$
EM	$\sigma^2 \left\{ \frac{\boldsymbol{\varepsilon}_{21}^2}{\boldsymbol{n}_1} + \frac{\boldsymbol{\varepsilon}_{12}^2}{\boldsymbol{n}_2} + (quadratic\ in\ z_p) \right\}$
QEM	$\sigma^2 \left\{ \frac{\boldsymbol{\varepsilon}_{21}^2 \boldsymbol{\varepsilon}_{31}^2}{\boldsymbol{n}_1} + \frac{\boldsymbol{\varepsilon}_{12}^2 \boldsymbol{\varepsilon}_{32}^2}{\boldsymbol{n}_2} + \frac{\boldsymbol{\varepsilon}_{13}^2 \boldsymbol{\varepsilon}_{23}^2}{\boldsymbol{n}_3} + (quadratic\ in\ z_p) \right\}$

Distribution	Var($\hat{\alpha}_0$)
Lognormal - EM	$\alpha_0^2 \left\{ \exp\left[\frac{1}{k^2} \left(\frac{\boldsymbol{\varepsilon}_{21}^2}{\boldsymbol{n}_1} + \frac{\boldsymbol{\varepsilon}_{12}^2}{\boldsymbol{n}_2} \right) \right] - 1 \right\} \exp\left[\frac{1}{k^2} \left(\frac{\boldsymbol{\varepsilon}_{21}^2}{\boldsymbol{n}_1} + \frac{\boldsymbol{\varepsilon}_{12}^2}{\boldsymbol{n}_2} \right) \right]$
Lognormal - QEM	$\alpha_0^2 \left\{ \exp\left[\frac{1}{k^2} \left(\frac{\boldsymbol{\varepsilon}_{21}^2 \boldsymbol{\varepsilon}_{31}^2}{\boldsymbol{n}_1} + \frac{\boldsymbol{\varepsilon}_{12}^2 \boldsymbol{\varepsilon}_{32}^2}{\boldsymbol{n}_2} + \frac{\boldsymbol{\varepsilon}_{13}^2 \boldsymbol{\varepsilon}_{23}^2}{\boldsymbol{n}_3} \right) \right] - 1 \right\} \exp\left[\frac{1}{k^2} \left(\frac{\boldsymbol{\varepsilon}_{21}^2 \boldsymbol{\varepsilon}_{31}^2}{\boldsymbol{n}_1} + \frac{\boldsymbol{\varepsilon}_{12}^2 \boldsymbol{\varepsilon}_{32}^2}{\boldsymbol{n}_2} + \frac{\boldsymbol{\varepsilon}_{13}^2 \boldsymbol{\varepsilon}_{23}^2}{\boldsymbol{n}_3} \right) \right]$

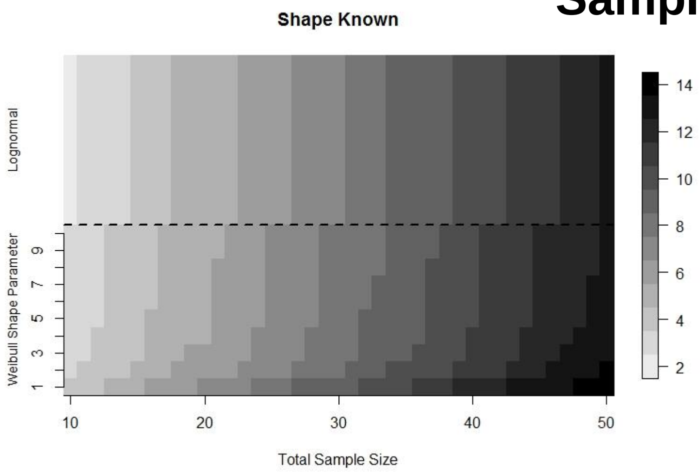
Test Plan Comparison – EM

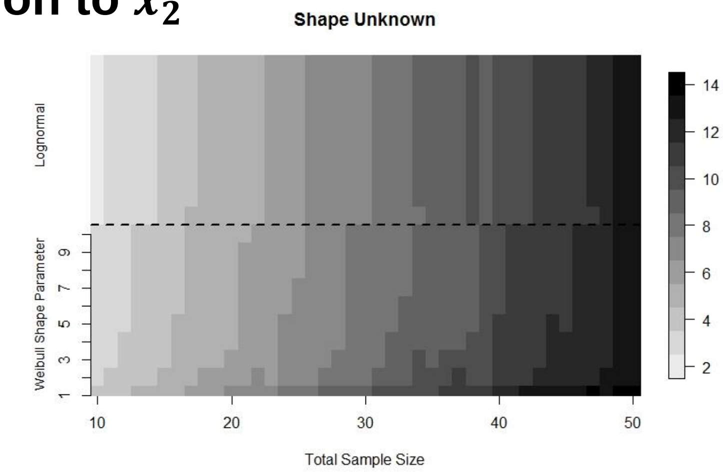
Sample allocation to x_2



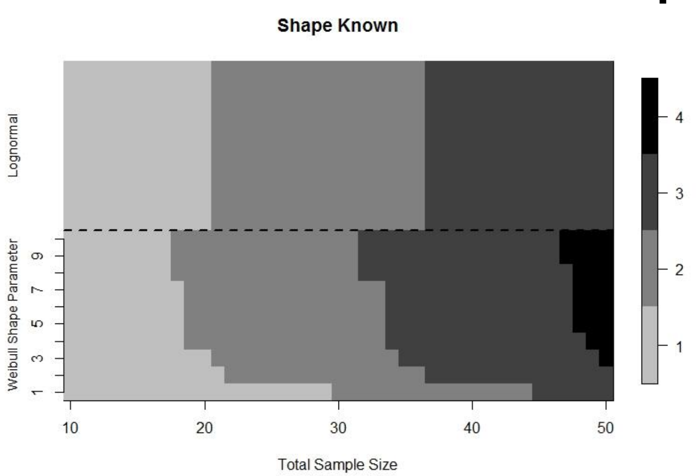
Test Plan Comparison – QEM

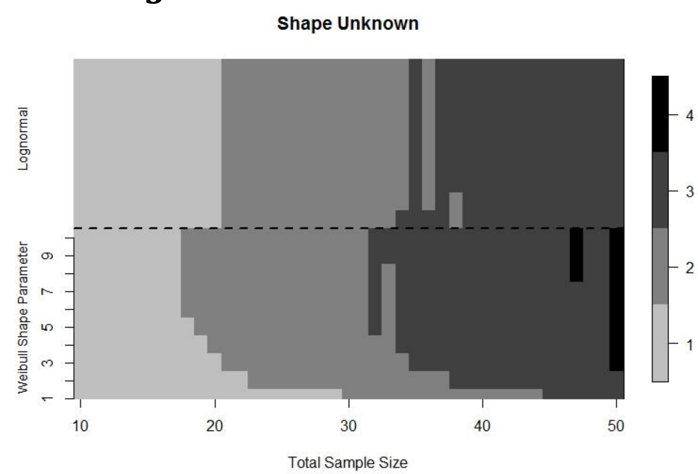
Sample allocation to x_2





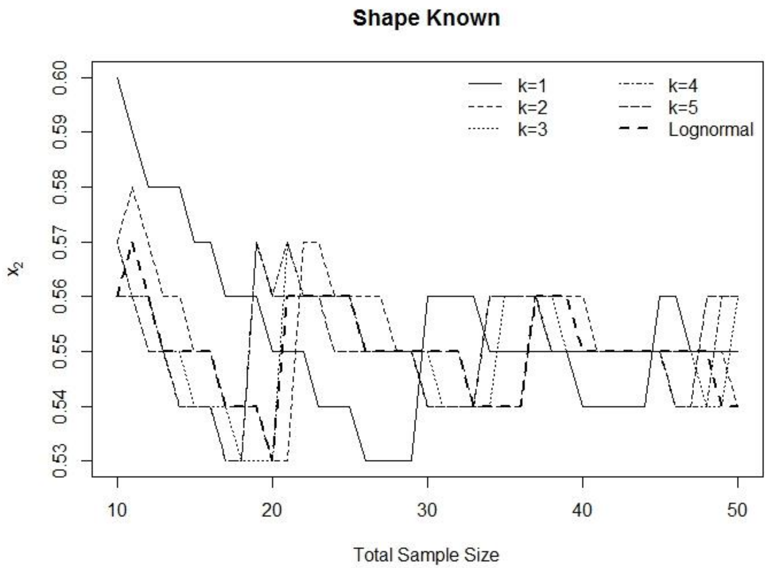
Sample allocation to x_3

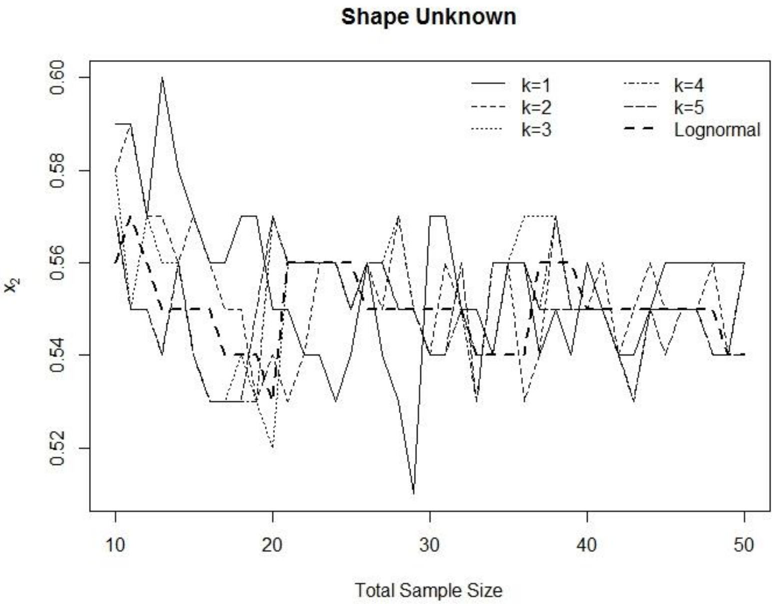




Test Plan Comparison – QEM

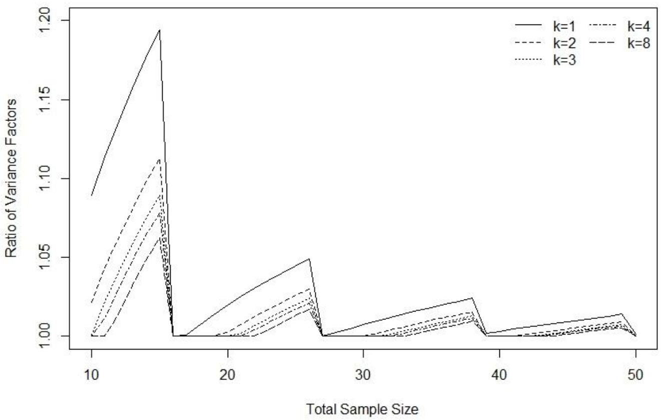
Location of x_2

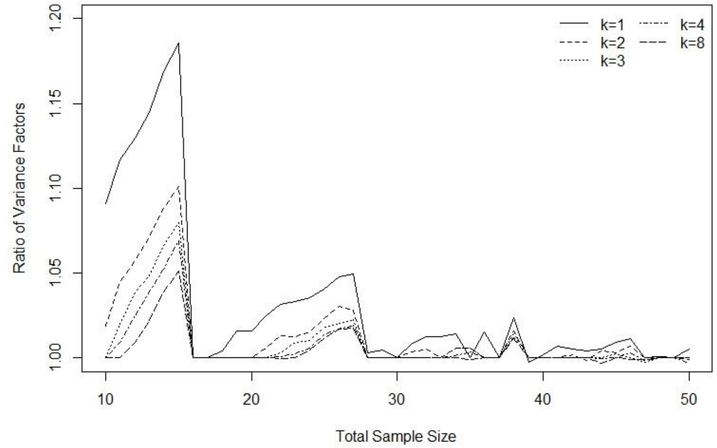




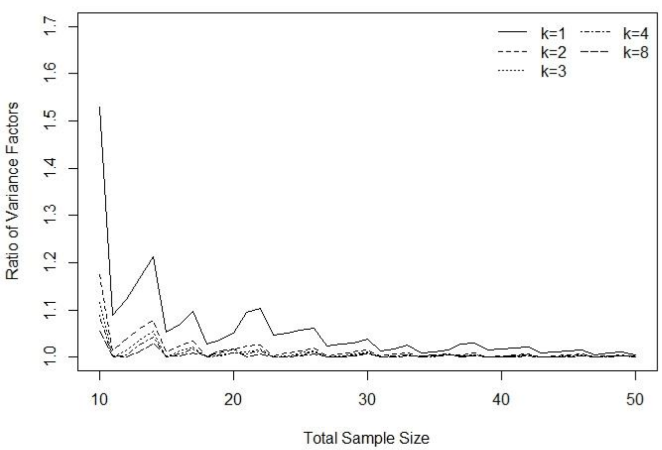
Test Plan Comparisons – Variance Ratios

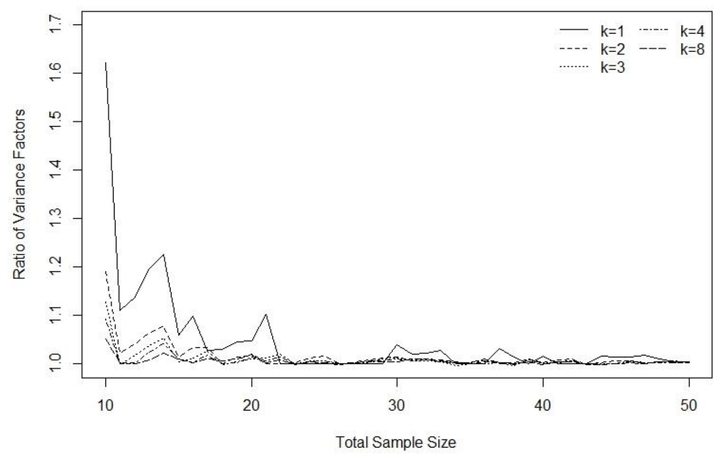
EM





QEM

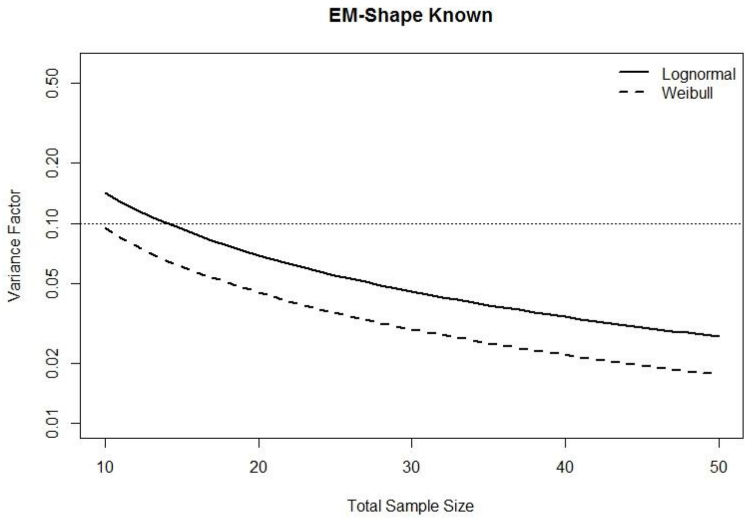


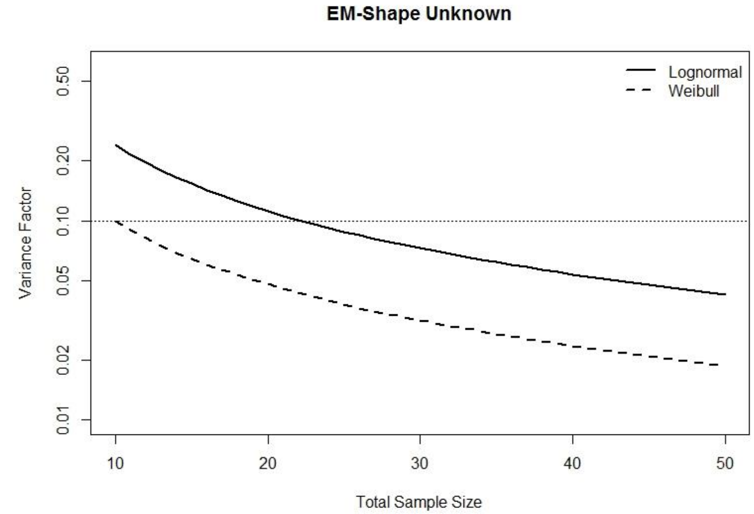


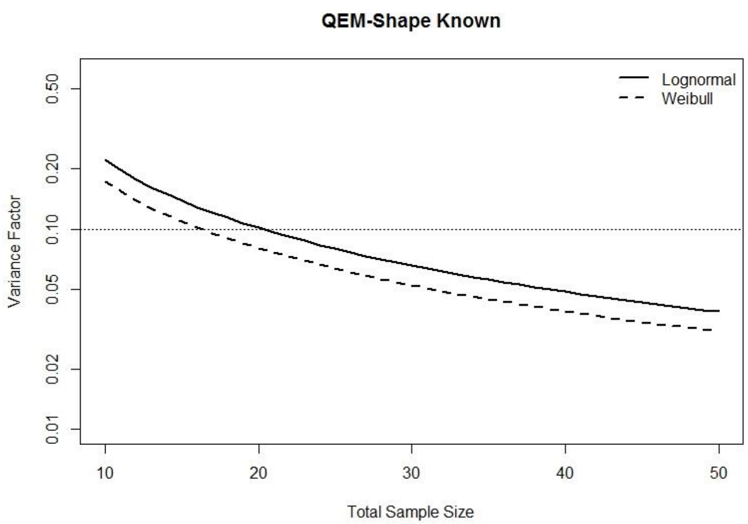
Sample Size Determination

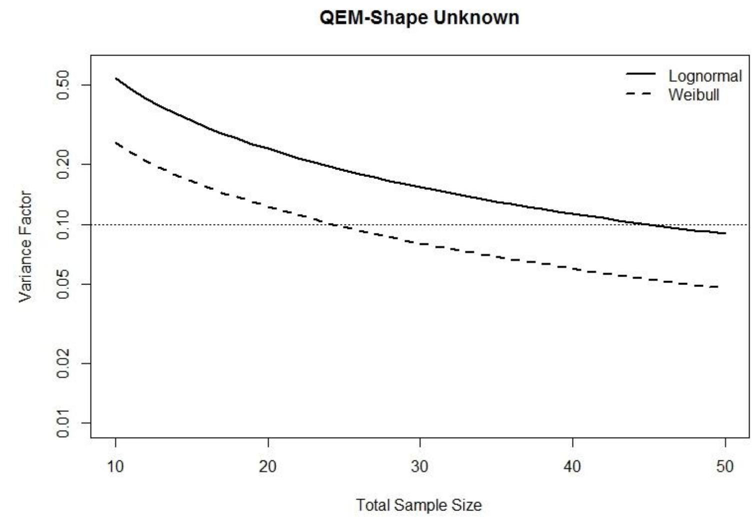
- Very little in literature on how to determine necessary sample size to achieve desired precision in optimal ALT design.
- Can easily be obtained by "inverting" optimal design.
  - Given a method for creating an optimal design, what is the minimum sample size needed to drop below a given threshold of precision.
- Will depend on some knowledge of shape parameter.

Sample Size Determination









Summary

- There is a clear distinction between test plans based on either methodology.
  - Greatest discrepancy for extremely small sample sizes and highly skewed distribution (roughly 20-60% difference).
  - Two methods start to be more consistent for sample sizes around 20 or above.
- Test plans seem to be relatively robust to lack of knowledge regarding shape parameter.
- Smart use of computational tools and techniques can help deal with complicated mathematics.

Future Research

- Allow for the possibility of censoring in the data.
  - Progress has already been made in deriving suitable estimator for Type I censoring under Weibull with known shape parameter.
- Consider other areas for generalization.
  - Multiple acceleration factors
  - Competing risks
  - More complex models and distributions
- Continue to refine and develop procedure as new situations arise.

Thank You!

Appendix

Asymptotic Variance Expression Derivation

- Let $u_{jk} = \left.\frac{\partial \mu_j}{\partial \theta_k}\right|_{x_j}$ and $z_{ij} = \frac{y_{ij} - \mu_j(x_j;\theta_1,\theta_2)}{\sigma}$.
- Further define,
  - $h_1 = \mathrm{E_Z}\left[-\frac{d^2}{dz^2}\ln\{\phi(z)\}\right]$
  - $h_2 = \mathrm{E_Z}\left[-z\frac{d^2}{dz^2}\ln\{\phi(z)\}\right]$
  - $h_3 = \mathrm{E_Z}\left[-z^2\frac{d^2}{dz^2}\ln\{\phi(z)\}\right]$
  - $\mathbf{H} = \begin{bmatrix} h_1 & h_2 \\ h_2 & h_3 + 1 \end{bmatrix}$

Asymptotic Variance Expression Derivation

- Let $\boldsymbol{I}_\theta$ be the Fisher information matrix.
- Define $\boldsymbol{U}_{jj'} = \begin{bmatrix} u_{j1} & u_{j'1} \\ u_{j2} & u_{j'2} \end{bmatrix}$ and

$$AVar(\hat{y}_p) = [u_{01} \quad u_{02} \quad z_p]\boldsymbol{I}_\theta^{-1}[u_{01} \quad u_{02} \quad z_p]^T$$

- It follows that
  - $|\boldsymbol{I}_\theta| = \frac{h_1}{\sigma^6} N n_1 n_2 |\boldsymbol{U}_{12}|^2 |\boldsymbol{H}|$
  - $AVar(\hat{y}_p) = \frac{\sigma^2}{h_1}\left\{\frac{1}{n_2}\left(\frac{|\boldsymbol{U}_{01}|}{|\boldsymbol{U}_{12}|}\right)^2 + \frac{1}{n_1}\left(\frac{|\boldsymbol{U}_{02}|}{|\boldsymbol{U}_{12}|}\right)^2 + \frac{h_1^2}{N|\boldsymbol{H}|}\left[z_p - \frac{h_2}{h_1}\left(\frac{|\boldsymbol{U}_{02}|}{|\boldsymbol{U}_{12}|} - \frac{|\boldsymbol{U}_{01}|}{|\boldsymbol{U}_{12}|}\right)\right]^2\right\}$
  - For three parameters, it follows from similar reasoning that
  - $AVar(\hat{y}_p) = \frac{\sigma^2}{h_1}\left\{\frac{1}{n_2}\left(\frac{|\boldsymbol{U}_{013}|}{|\boldsymbol{U}_{123}|}\right)^2 + \frac{1}{n_1}\left(\frac{|\boldsymbol{U}_{023}|}{|\boldsymbol{U}_{123}|}\right)^2 + \frac{1}{n_3}\left(\frac{|\boldsymbol{U}_{012}|}{|\boldsymbol{U}_{123}|}\right)^2 + \frac{h_1^2}{N|\boldsymbol{H}|}\left[z_p - \frac{h_2}{h_1}\left(\frac{|\boldsymbol{U}_{012}|}{|\boldsymbol{U}_{123}|} + \frac{|\boldsymbol{U}_{023}|}{|\boldsymbol{U}_{123}|} - \frac{|\boldsymbol{U}_{013}|}{|\boldsymbol{U}_{123}|}\right)\right]^2\right\}$